Fantastic Fractals Fabricated by Frantic Fiddling Fingers

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Field of Interest: Mathematics and Biology

Brief Overview:
Mentees will be making crafts and drawing strange pictures to see what fractals look like and how they may arise in nature.

Agenda:

- Introduction (5 min)
- Module 1: Symmetric Snowflakes (15-20 min)
- Module 2: Trippy Triangles (10-15 min)
- Module 3: Bumbling Blobs (15-20 min)
- Conclusion (5 min)

Main Teaching Goals/Key Terms:

- Rotational symmetry
- Translational symmetry
- Reflectional symmetry
- Lines of symmetry
- Fractals
- Self-similarity
- Recursion
- Cellular Automata
    - Cells
    - Initial state
    - Generation
    - Rules
- Emergent behavior

Background for Mentors

Module 1

- Rotational symmetry
- Translational symmetry
- Reflectional symmetry
- Lines of symmetry

Mathematicians, for better or for worse, care deeply about patterns. One type of pattern that arises in geometry is **symmetry**, which is some way of mapping an object to itself while preserving its structure. Natural examples include rotating an object or reflecting it over a mirror.

More rigorously, **rotational symmetry** is when a given object can be rotated some amount and look the same. Examples include circles, since rotating by any amount keeps it the same, squares, since rotating 90 degrees makes it look the same, or more generally any regular polygon. Some examples in nature include starfish or worms.

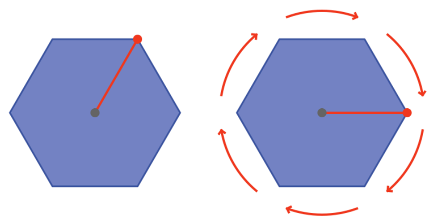

Figure 1: Rotational symmetry of hexagon

Translational symmetry is when something can be moved or shifted, but not rotated, and still look the same. Note this definition implies it must be some infinitely extending pattern, otherwise it would have edges that wouldn't align! Note, we don't really interact with infinite patterns in the real world, but many patterns we do see can be extended infinitely. Some examples include a brick wall pattern or a beehive pattern.

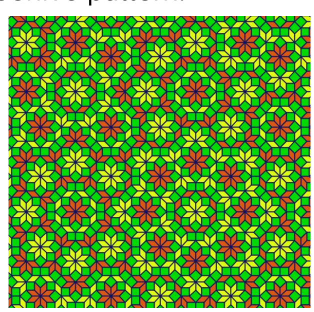

Figure 2. Tiling that has translational symmetry by moving up and to the right

Reflectional symmetry is when an object can be mirrored across a line and look the same. Such a line is called a **line of symmetry**. Humans and many mammals are bilateral, which means there is a line of symmetry down our center of which we are roughly symmetric, with a left and a right side.

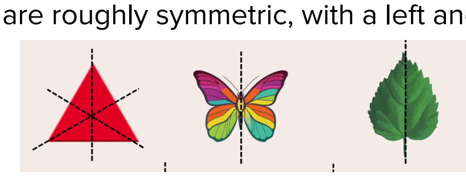

Figure 3: Reflectional symmetry of some objects

We often find symmetry in nature since it's evolutionarily advantageous to have duplicate body parts, such as two legs or two eyes. For primitive creatures like worms, you can think that maybe it would require less DNA to come up with a cylinder structure than it would something less symmetric.

Symmetry is also very handy in architecture. Bridges or beehives make use of triangles and hexagons as they are very sturdy and evenly balance forces. Arches and bridges have reflectional symmetry to keep forces such as gravity or payload even over the structure.

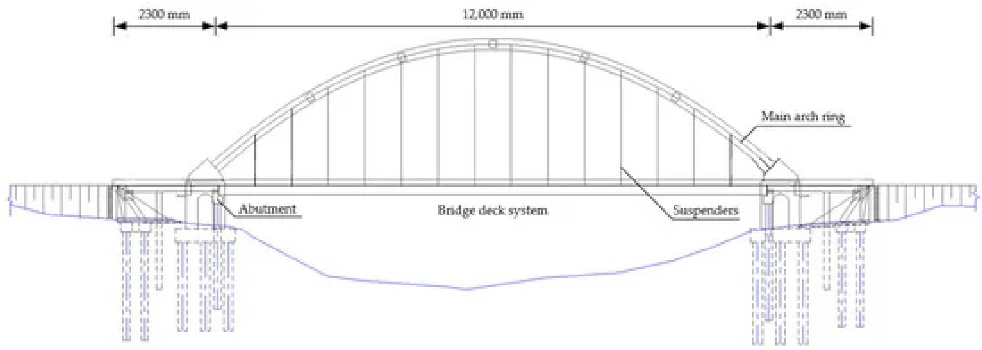

Figure 4: Reflectional symmetry of a bridge

Module 2

- Fractals
- Self-similarity
- Recursion

There are some objects that seem to exhibit some symmetry that isn't entirely described by our vocabulary in module 1. For instance, look at romanesco broccoli.

Figure 1: Romanesco broccoli

Although one can maybe find reflectional or rotational symmetry, we see another pattern: if we break apart the florets, they look exactly the same to the whole broccoli.

As the mathematician you are, you want to isolate what this type of symmetry is and how it is different. This will lead you to define the concept of **self-similarity**, where properties at finer and finer scales mimic those of the original object as a whole. This seems to describe the phenomena we originally saw.

We then turn our attention to a seemingly unrelated question: how long is the coast of Britain? Maybe you first use a 100 mile ruler and get one number. You try again with a 5 mile ruler and get a significantly higher result. Puzzled, you try again and again with smaller and smaller rulers giving you this data:

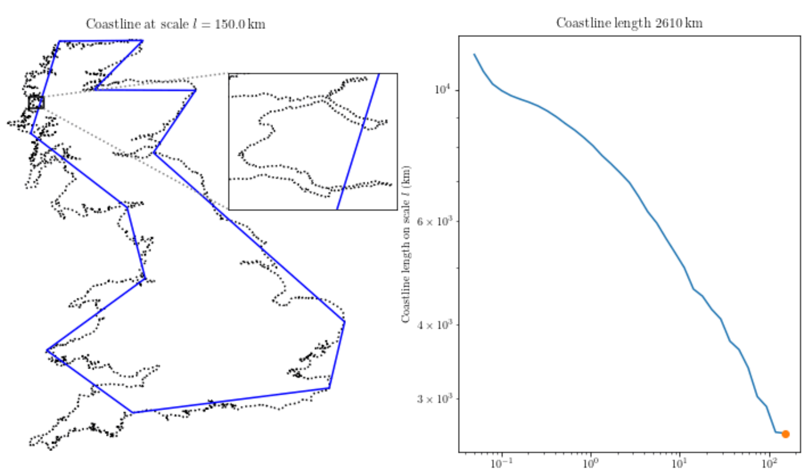

Figure 2: [The coastline paradox illustrated]

The coastline of Britain is so *rough*, with such fine detail, that we measure its perimeter to be infinite. What's going on here?

This is the essence of **fractals.** Although they don't have a widely agreed upon definition, they have to do with objects that have complex detail at any scale. That is, no matter how far you zoom, there is still complex behavior going on. Look at this fractal below:

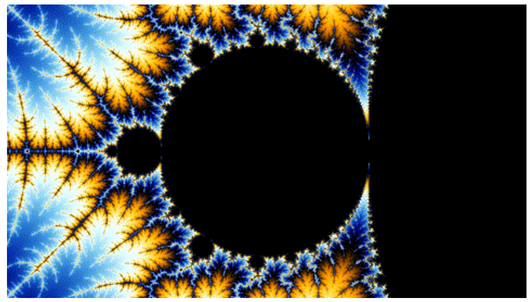

Figure 3: Zooming in on the Mandelbrot set

Contrast this with a circle: if you zoom in on the boundary, it just looks like a line. This fractal, called the *Mandelbrot set*, has these repeating bulbs and strange branches at every scale.

Although many fractals studied are usually self-similar, this is not always the case. **Fractals have to do with roughness, NOT self-similarity**. So in our case, romanesco broccoli is a self-similar fractal, but the coast of Britain, which has infinite detail as we zoom in, is clearly not self-similar.

The reason a lot of the fractals people study *are* self-similar, is because they are defined **recursively**. That is, we apply a procedure repeatedly on finer scales to induce some sort of self-similarity and create that complex detail. For instance, maybe we break a red triangle up into four smaller triangles, coloring the middle one black. We do this again for the remaining red triangles. Repeating this indefinitely gives us something like the Sierpinski triangle from the activity of this module.

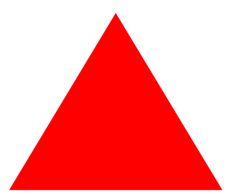

Figure 4: Recursive steps to build the Sierpinski triangle

Module 3

- Cellular Automata
  - Cells
  - Initial state
  - Generation
  - Rules
- Emergent behavior

Referring back to the romanesco broccoli example, how do these complex structures arise? How complicated do biological mechanisms have to be to cause a pattern like this to occur?

Indeed, we can use **cellular automata**, a discrete computational model, as a crude approximation for how these biological systems work. Cellular automata consist of several discrete **cells** which can be thought of as squares in a grid, each with a **state** that changes with time (such as dead or alive if modeling bacteria, or perhaps alive, dirt, and on fire to model forest fires). We use discrete, fixed, time intervals between states, for which we apply a given set of **rules** that determine the next state of the cell based on its neighbors. Each timestep creates a new **generation**, which is the collective state of the entire model at that fixed time. You can think of this like a generation of bacteria, where within every timestep all the bacteria attempt to reproduce and die, with some new bacteria possibly taking its place. I recommend you play around with this website that uses cellular automata to simulate forest fires to see what complicated behavior may arise!

Figure 1: Conway's game of life

As seen above, very complicated behavior, which we call **emergent behavior**, can occur from these simple rules. Looking at the activity for this module, with just 8 simple rules, we are able to generate the Sierpinski triangle from module 2! In fact, although we use rule 22, the variant rule 110 has been shown to be Turing complete, which means it can theoretically be capable of doing anything a modern computer can. This phenomena can explain why the romanesco broccoli is able to have such fine structure, or why ferns and other plants have self-similarity.

In our activity in module 3, the cellular automata model is of a 1 dimensional line, with each timestep represented spatially through the vertical direction. That is, each timestep, and hence generation, is just a different row. Seeing time as a spatial dimension is confusing, but plants behave exactly like this! The base of a fern is younger while the upper tip of the fern is older and the outer rings of a tree are younger than the central ones. Similarly, the bottom rows of our activity are "younger" and the upper rows are "older".

	 **Figure 2:** Barnsley fern, a fractal inspired by natured

Introduction

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Concepts to Introduce

- Symmetry
  - Some sort of pattern that an object has. What are different types of symmetry that we encounter daily?
- Fractals
  - Objects that experience symmetry with themselves at smaller and smaller scales
- Cellular automata
  - A computational model that follows simple rules yet complex behavior arises.

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Questions to Pique Interest

- How do we classify patterns we see on objects in our daily lives?
- How do these patterns come about? How are they useful?
  - For instance, why are arches bilateral and why are beehives hexagonal?
- How do fractals arise in nature?

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Scientists, Current and Past Events

- A paper using fractals to model the spread of COVID-19
- Benoit B. Mandelbrot solidified fractal geometry as a mathematical field, coining the term fractal and strongly developing the theory
- Fractal image coding is an image compression algorithm that uses fractals since with very little data they can come up with seemingly random patterns.
- Popularity as a symbol for math grew with personal computers where they were used as screensavers to show off how powerful they were.

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Careers and Applications

- Mathematical researchers and researchers in general take nebulous questions and slowly tease apart specifics to study further.
- Theoretical and computational biologists are interested in cellular automata as they are simple models that exhibit some of the same complex behavior we see in the real world.
- Can be used to detect cancer since parts of the human body like veins are fractal like while cancerous cells aren't.

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Module 1: Symmetric Snowflakes

Learn different types of symmetry through a snowflake build. Although every snowflake is unique, they still have a common trait: symmetry! Each mentee will make their own snowflake and identify each of its different types of symmetry.

Teaching Goals	Materials
1. **Rotational symmetry:** When an object can be rotated by some amount and look the same. 2. **Translational symmetry:** When a pattern can be moved without rotating and looks the same. 3. **Reflectional symmetry:** When an object can be divided across a line, with one side as the mirror image of the other. 4. **Lines of symmetry:** The line(s) over which an object experiences reflectional symmetry	• Origami paper • Scissors • Markers

Different Methods for Teaching

Real-Life Examples: Try to prompt mentees to identify objects in and out of the classroom with different types of symmetry! For mentees already familiar with symmetry, you can explore why symmetry appears in our lives. For example, talk about why mammals tend to be bilateral, or how the rotational symmetry of hexagons make beehives very strong. You can also draw a wealth of examples from architecture.

Procedure

1. Fold paper along the diagonal.
2. Fold the triangle in half, dividing the long edge
3. Take one corner along the long edge and fold it past the center.
4. Fold the other side over it.
5. Fold in half with everything in the middle, and cut the top leaving you with a triangle.
6. Perform various cuts, provided you leave the folded edge sufficiently structurally sound.
7. Unfold.

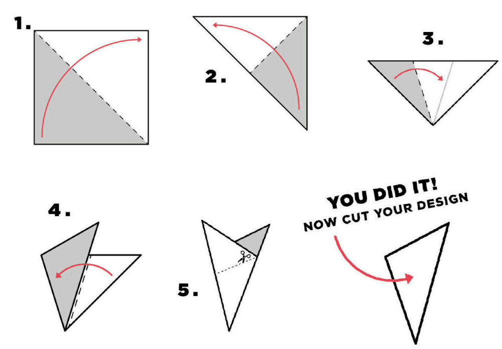
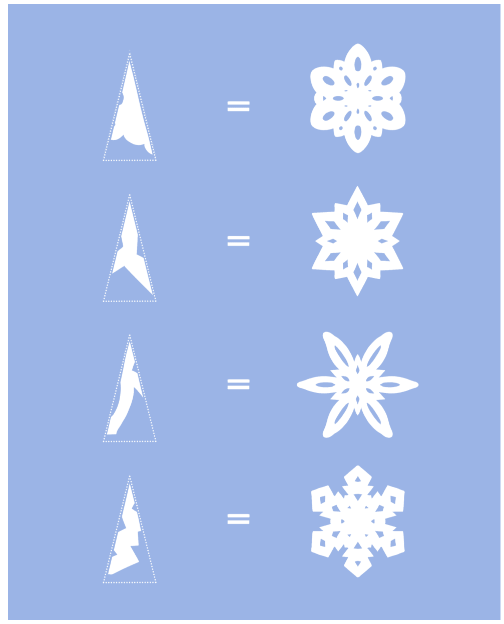


Figure 1: Step by step on folding

	 **Figure 2:** Some potential snowflake patterns for inspo

Classroom Notes

Be sure to clean up all the paper scraps! You can also have a mentor make two identical snowflakes to illustrate how when you rotate by some amount, the snowflakes look the same.

Module 2: Trippy Triangles

Mentees will be using an iterative procedure to build their own approximation of a fractal.

Teaching Goals	Materials
1. **Fractals:** A geometric shape that contains detail at arbitrarily small scales that is self-similar. 2. **Self-similarity:** A type of symmetry when properties, usually its shape, approximately repeat on smaller and smaller scales. 3. **Recursion:** The repeated application of a procedure.	• Images of fractals • Printed templates (triangle or pop out card) • Markers • Scissors

Different Methods for Teaching

1. **Building Up:** To help introduce fractals, show the examples found in nature. There seems to be some sort of finer symmetry than what was taught in module 1. What is that symmetry? What properties does it have? Mathematicians often observe phenomena and try to extrapolate the most fundamental aspect of it.
2. **Tying the activity back to teaching goals:**
    a. A notable part of this activity is putting the triangles together or disassembling the card. The structures look identical as you build them up or break them apart, just on a different scale, illustrating **self-similarity.**
    b. At each step of constructing the fractals, the mentees are repeating some steps of the procedure, just at a smaller scale which connects to recursion.

Sierpinski triangles procedure

1. Draw lines between the midpoints of the triangle to form a central triangle and fill it in.
2. Keep doing this for the blank triangles that remain! Repeat as many times as desired.
3. Cut out the triangle and have mentees write their names on the back.
4. If time permits, cut out the big triangle and combine mentees triangles to form a BIG Sierpinski triangle.

Cutout card procedure

1. Fold the template in half.
2. Cut along the largest dotted line labeled “1.” and create a fold over the newly made cut
3. Unfold and invert the fold inward.
4. Fold in half again and repeat with the shorter

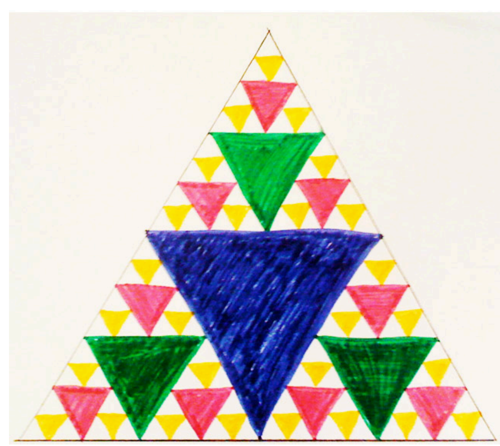


Figure 1: End result of Sierpinski triangle procedure

dotted lines! Stop until the desired detail is reached.

5. Optional: Have mentees cut their bigger card into three smaller, identical cards to emphasize self-similarity.

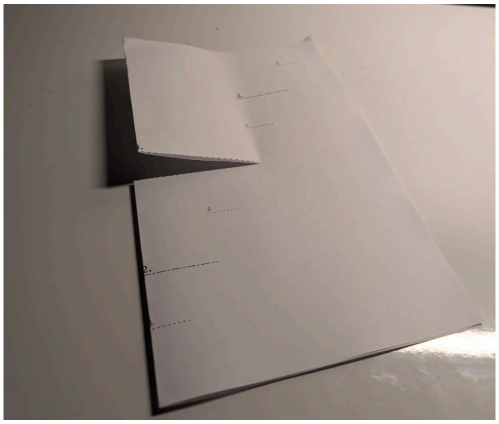


Figure 2: Cut the card over the longest dotted line

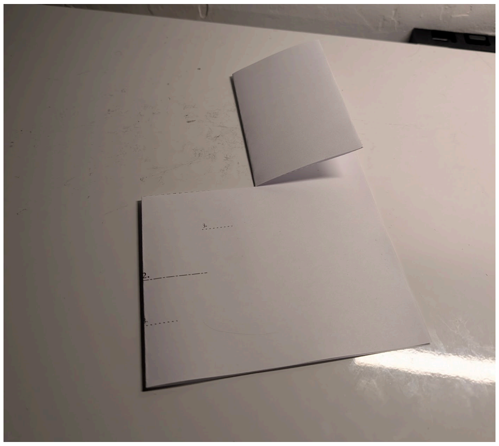


Figure 3: Bend the top portion

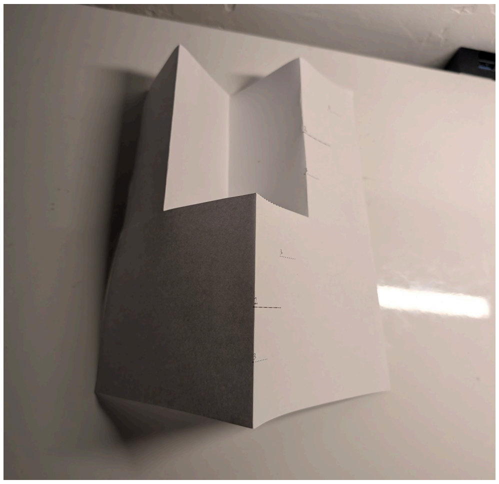


Figure 4: Unfold, and invert the top part as shown

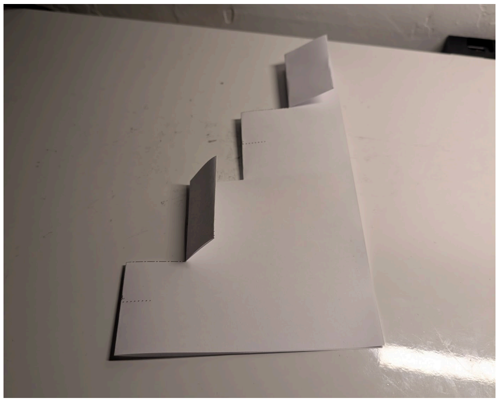


Figure 5: Fold in half again, this time cutting the lines labeled with “2.”

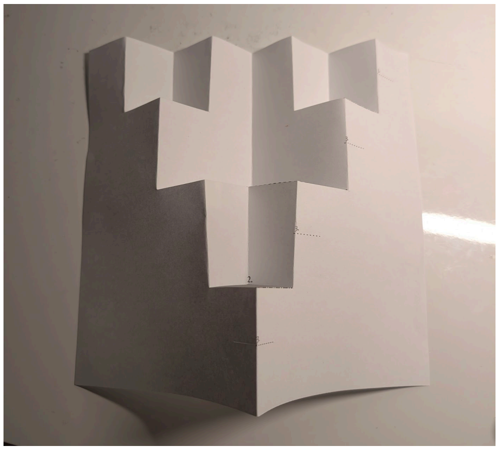


Figure 6: Unfold and invert

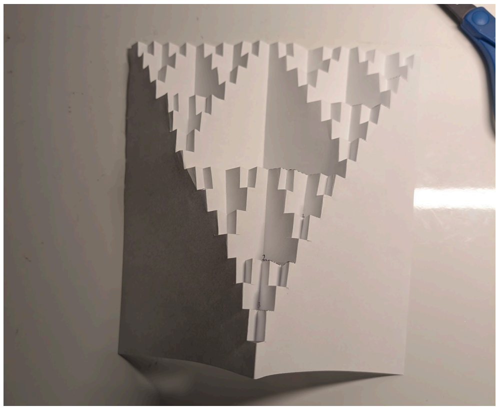
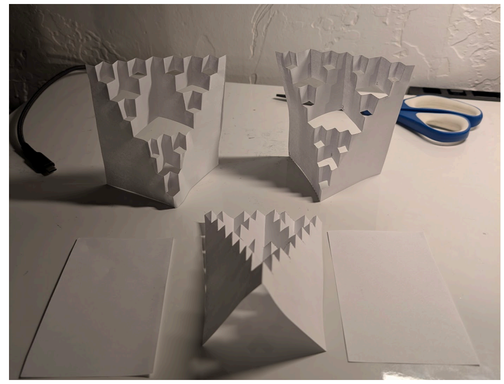


Figure 7: Repeat!

	 **Figure 8:** Cutting it up into smaller parts

Classroom Notes

The cutout card activity should be more engaging for older mentees, but feel free to do the Sierpinski triangle to save time.

Module 3: Bumbling blobs

Mentees will learn how fractals and self-similar structures may arise through simulating cellular automata. This can be done as a class or in small groups.

Teaching Goals

1. **Cellular Automata:** A discrete model of computation where cells are born and die over discrete time steps.
    a. **Cells:** Whether a blob is present or not (e.g. our blobs are black)
    b. **Initial state:** The state of the blobs at the beginning (e.g. the top row)
    c. **Generation:** The state of the blobs after a fixed timestep (e.g. a single row)
    d. **Rules:** Rules that govern how the blobs grow and die
2. **Emergent behavior:** When complex patterns and phenomena arise from simple behaviors and rules.

Materials

- Printed grid template or grid easel
- Printed out rules
- Writing utensils
- Template tetromino

Different Methods for Teaching

It may be confusing to think about time as a spatial direction (older generations on top and new generations on bottom). You can tie this back to the example of ferns with younger leaves at the base and older leaves at the tip, or also how the outer rings of a tree are younger than the central ones.

Tying back to the teaching goals:

a. Observe that these simple rules created a very complex result. It seems then that there might be simple rules in nature that create complicated **emergent behaviors** that we can observe, such as the broccoli or the snail shells.

Full classroom procedure

1. Start with a single dot as in the individual procedure.
2. Put the template tetromino such that the top of the T-shape is on the first row and the single square is over the second row. (Look at figure 2 for instance)
3. Go over each cell in that row, and use the rules to determine if they need to be shaded based on what blobs are “captured” or contained at the top of the template. Based on what colors are at the top of the template, find which rule

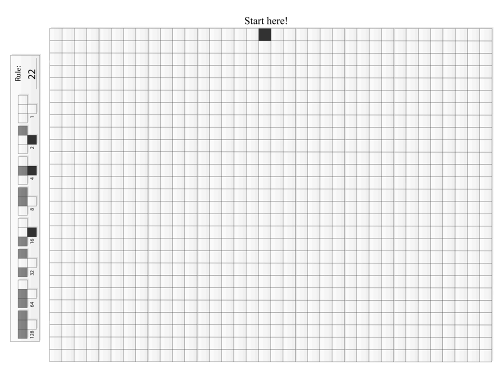


Figure 1: The initial state

applies, and shade the bottom square accordingly (look at figure 2 for an example). Note: consider the boundary as being white.

4. Continue doing this for each row until you finish or run out of time.

Individual/pair procedure

1. First do a few examples as a class to get started so that mentees can understand the rules.
2. Picture a T-shape over the top row (such as in figure 2). Shade in the box at the bottom of the shape based on the rules and what blobs are contained at the top of the T-shape. Note: we consider the boundary as being white.
3. Continue doing this for every row, going from the top to the bottom.

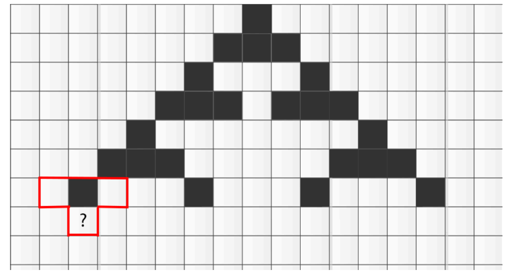


Figure 2: As the top three squares above the question mark only contain one blob in the middle, we identify that this is rule 4 and so we put a blob on the question mark.

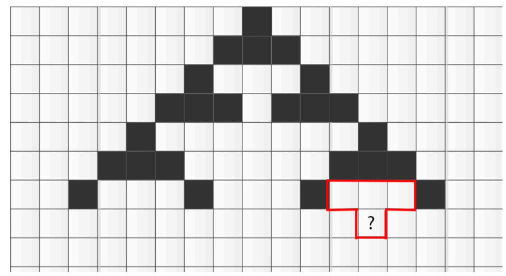


Figure 3: The top three squares are white which is rule 1, so we don't put a blob on the question mark.

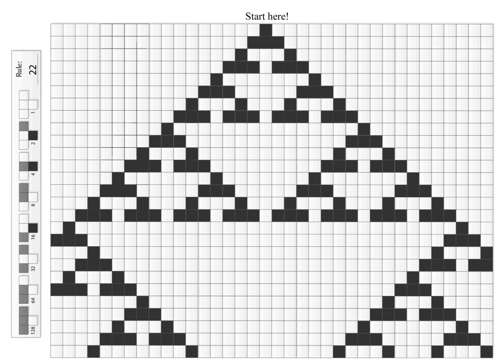


Figure 4: Final result! This is the Sierpinski from the last module!

Classroom Notes

It's hard to get started since the rules are complicated. If you do the individual/pair activity, perhaps walk through the first two rows together.

Conclusion

In this lesson, mentees explore different patterns objects may exhibit, and pay special attention to self-similarity. We then explored cellular automata to model how self-similarity may arise in nature.

References

- Fractivities, Fractal Foundation. https://fractalfoundation.org/resources/fractivities/

Summary Materials Table

Material	Amount per Site	Expected $$	Vendor (or online link)
Origami paper	1 per student	11	Amazon
Paper	1-2 per student	23	Amazon
Grid easel pads	1 per classroom	50	Amazon
Colored prints	4 per classroom	0	Public service center